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Higher Order Effects of the Phase Change within a Drift Space

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Introduction

In tracking codes of accelerator physics, changes of the "longitudinal coordinate" in non-accelerating elements are typically considered only up to the linear order of the "longitudinal momentum". This is a good approximation for a machine with a large relativistic gamma. For a low energy machine, however, the higher order effects may not be negligible and the TraceWin is in fact taking into account the higher order effects. Figure 1 shows the average phase of a bunch with respect to the reference particle within the ESS MEBT, calculated by the TraceWin. The energy of the reference particle is the same as the average energy of the bunch and it moves on the z-axis. Because the particles in the bunch have non-zero transverse divergences, their speed projected on the z-axis is smaller than the total speed. This causes a lag of a bunch in a drift space and explains the initial increase of the phase in Figure 1. (As seen in the following, there is a high order effect related to the longitudinal divergence as well.) The purpose of this note is to derive expressions for these higher order effects. For simplicity, we only consider a drift space in this note.

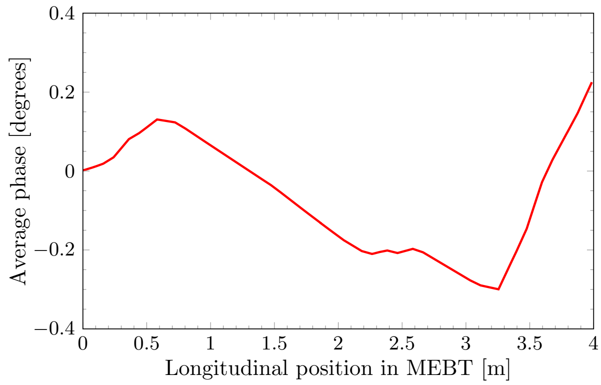

Figure 1: The average phase of a bunch with respect to the reference particle within the ESS MEBT.

Review of the First Order Expression

In this section, we set-up the problem and review the commonly used first order expression. The TraceWin uses the phase ϕ (equivalent to the arrival time difference δ_t) as the longitudinal coordinate and the fractional momentum $\delta_p \equiv (p - p_0)/p_0$ as the longitudinal momentum, where p and p_0 are the total momenta of the test and reference particles. The choice of the longitudinal coordinate and momentum depends on a code and is in fact not a simple subject. As an example, neither pair, (ϕ, δ_p) or (δ_t, δ_p), is actually canonically conjugate each other. Nevertheless, this is beyond the scope of this note and the author is planning to write a separate note on this subject. In the following, we follow the convention of the TraceWin and use the phase and fractional momentum.

Calculating the change of the arrival time over a drift space with a length Δs, $\Delta\delta_t$, is a problem of elementary level physics and the solution is

$$\Delta\delta_t = \frac{\Delta s}{c\beta_z} - \frac{\Delta s}{c\beta_0}\,, \tag{1}$$

where c is the speed of light, β_z is the z component of the relativistic beta of the test particle, and β_0 is the total relativistic beta of the reference particle. The change of the phase over the same drift space, $\Delta\phi$, is given by simply multiplying 2π and the frequency f to $\Delta\delta_t$. If we are only interested in the first order relation between $\Delta\phi$ and δ_p, we can assume $\beta_z \sim \beta$, where β is the total relativistic beta of the test particle, (paraxial approximation) and then $\Delta\phi$ is given by

$$\Delta\phi \sim -\frac{2\pi\Delta s}{\beta_0\lambda}\frac{\beta-\beta_0}{\beta_0} \sim -\frac{2\pi\Delta s}{\beta_0\lambda}\frac{1}{\gamma_0^2}\delta_p\,, \tag{2}$$

where $\lambda = c/f$ is the wavelength and we used the following first order relation between β and p:

$$\frac{\delta\beta}{\beta} \sim \frac{1}{\gamma^2}\frac{\delta p}{p}\,. \tag{3}$$

Equation (2) is the commonly used first order relation of $\Delta\phi$ and δ_p.

Full Order Expression

In this section, we derive the full order expression of $\Delta\phi$. Beyond the first order, $\Delta\phi$ depends on not only the longitudinal momentum but also the transverse ones. As usual in accelerator physics, our transverse momenta are the angles x' and y' which satisfy

$$x' = \frac{dx}{ds} = \frac{\beta_x}{\beta_z} \tag{4}$$

$$y' = \frac{dy}{ds} = \frac{\beta_y}{\beta_z}\,, \tag{5}$$

where s is the path length of the reference particle and β_x (β_y) is the x (y) component of the relativistic beta. Back to Equation (1), we rewrite the equation in the following way

$$\Delta\phi = \frac{2\pi\Delta s}{\beta_0\lambda}\left(\frac{\beta}{\beta_z}\frac{\beta_0}{\beta} - 1\right). \tag{6}$$

In this expression, we can already grasp the effects of the higher order terms. The term β/β_z is describing that the non-zero transverse divergence makes the z component of the velocity smaller than the total velocity and thus causes a lag with respect to the reference particle. The term β_0/β is the ratio of the velocities of the reference and test particles and simply describing that the phase reduces (increases) if a test particle is faster (slower) than the reference particle. At this point, our problem is reduced to express β and β_z with momenta x', y', and δ_p. The ratio β/β_z can be easily written with the angles x' and y':

$$\frac{\beta}{\beta_z} = \sqrt{1 + x'^2 + y'^2}\,. \tag{7}$$

To rewrite the ratio β_0/β with δ_p, first we express β with p, p_0 and the energy E in the following way:

$$\beta = \frac{cp}{E} = \frac{p}{p_0}\frac{cp_0}{E}\,. \tag{8}$$

With p_0 and δ_p, p and E can be written as

$$\frac{p}{p_0} = 1 + \delta_p \tag{9}$$

$$\frac{E}{cp_0} = \sqrt{(1+\delta_p)^2 + 1/(\gamma_0\beta_0)^2}\,. \tag{10}$$

By substituting these two equations into Equation (8), we get the following relation between β and β_0 expressed with δ_p:

$$\beta = \frac{1+\delta_p}{\sqrt{(1+\delta_p)^2 + 1/(\gamma_0\beta_0)^2}} = \frac{\beta_0}{\sqrt{1 - 1/\gamma_0^2 + 1/[\gamma_0(1+\delta_p)]^2}}\,. \tag{11}$$

In the second step, we used $\beta_0^2 = 1 - 1/\gamma_0^2$ to show explicitly that β becomes β_0 when $\delta_p = 0$. Finally, substituting Equations (7) and (11) into Equation (6), we get this full order expression:

$$\Delta\phi = \frac{2\pi\Delta s}{\beta_0\lambda}\left\{\sqrt{(1+x'^2+y'^2)\left[1 - \frac{1}{\gamma_0^2} + \frac{1}{\gamma_0^2(1+\delta_p)^2}\right]} - 1\right\}. \tag{12}$$

Please note that, in addition to the terms related to the transverse divergences, there is a higher order effect related to δ_p. This will be clearer when we see the expanded expression in the next section.

Second Order Expression

The previous section derived the full order expression of $\Delta\phi$. On the other hand, the expanded form up to the second order is often more than enough and also useful for some purposes. Thus, in this section, we perform an expansion of Equation (12) and derive the second order expression of $\Delta\phi$. First, we introduce a function $f(\delta_p)$ defined as

$$f(\delta_p) \equiv 1 - \frac{1}{\gamma_0^2} + \frac{1}{\gamma_0^2(1+\delta_p)^2}\,. \tag{13}$$

Then, our problem becomes to perform the Taylor expansion of $f(\delta_p)^{1/2}$ around $\delta_p = 0$ up to δ_p^2. The first and second derivatives of $f(\delta_p)^{1/2}$ are given by

$$\frac{d\left[f(\delta_p)^{1/2}\right]}{d\delta_p} = -\frac{1}{\gamma_0^2(1+\delta_p)^3} f(\delta_p)^{-1/2} \xrightarrow{\delta_p=0} -\frac{1}{\gamma_0^2} \tag{14}$$

$$\frac{d^2\left[f(\delta_p)^{1/2}\right]}{d\delta_p^2} = \frac{3}{\gamma_0^2(1+\delta_p)^4} f(\delta_p)^{-1/2} - \frac{1}{\gamma_0^4(1+\delta_p)^6} f(\delta_p)^{-3/2} \xrightarrow{\delta_p=0} \frac{3}{\gamma_0^2} - \frac{1}{\gamma_0^4} = \frac{3\beta_0^2}{\gamma_0^2} + \frac{2}{\gamma_0^4}\,. \tag{15}$$

The last step of Equation (15), where $(3/\gamma_0^2 - 1/\gamma_0^4)$ was rewritten as $(3\beta_0^2/\gamma_0^2 + 2/\gamma_0^4)$, was not necessary and just for a discussion later (and the author's preference to use the terms with the same signs). With Equations (14) and (15), the expansion of $f(\delta_p)^{1/2}$ is given by

$$f(\delta_p)^{1/2} = 1 - \frac{1}{\gamma_0^2}\delta_p + \left(\frac{3\beta_0^2}{2\gamma_0^2} + \frac{1}{\gamma_0^4}\right)\delta_p^2 + \cdots\,. \tag{16}$$

By using this equation in Equation (12) and combining with the expansion of $(1 + x'^2 + y'^2)^{1/2}$, we get

$$\Delta\phi \sim -\frac{2\pi\Delta s}{\beta_0\lambda}\left[\frac{1}{\gamma_0^2}\delta_p - \left(\frac{3\beta_0^2}{2\gamma_0^2} + \frac{1}{\gamma_0^4}\right)\delta_p^2 - \frac{1}{2}\left(x'^2 + y'^2\right)\right]. \tag{17}$$

Please note that $\Delta\phi$ is proportional to $2\pi\Delta s/(\beta_0\lambda)$ and this beta is one for the reference particle and not for the test particle. The expression directly derived in the Hamiltonian formalism is proportional to $2\pi\Delta s/(\beta\lambda)$, however, and in this case $\Delta\phi$ is given by

$$\Delta\phi \sim -\frac{2\pi\Delta s}{\beta\lambda}\left[\frac{1}{\gamma_0^2}\delta_p - \frac{3\beta_0^2}{2\gamma_0^2}\delta_p^2 - \frac{1}{2}\left(x'^2 + y'^2\right)\right]. \tag{18}$$

We can easily check that Equations (17) and (18) are equivalent up to δ_p^2 by using the following first order relation

$$\frac{\beta}{\beta_0} \sim 1 + \frac{1}{\gamma_0^2}\delta_p\,. \tag{19}$$

This shows that we need to be careful whether a parameter is either one for the test particle or reference particle when taking into account the higher order effect.

Discussion

The advantage of the second order expression, Equation (17), is to allow us to calculate the average phase of a bunch with the RMS values of momenta. Taking the average over the particles in a bunch on both side of Equation (17), we get

$$\langle\Delta\phi\rangle \sim -\frac{2\pi\Delta s}{\beta_0\lambda}\left[\frac{1}{\gamma_0^2}\langle\delta_p\rangle - \left(\frac{3\beta_0^2}{2\gamma_0^2} + \frac{1}{\gamma_0^4}\right)\langle\delta_p^2\rangle - \frac{1}{2}\langle x'^2 + y'^2\rangle\right], \tag{20}$$

where the triangular bracket denotes the average. For example in the ESS MEBT, as seen in Figure 1, the effect from the higher order terms reaches ~0.3 degrees and this is a significant fraction of one degree, which is the typically assumed accuracy for cavity phases in tracking simulations. If such a systematic effect from the higher order terms becomes an issue, the cavity phases can be adjusted in the tracking simulation by using Equation (20) and the RMS values of x', y', and δ_p from the envelope calculation. (Physically, this means to use the bunch as the reference instead of a particle.)

We conclude this note by looking at some numbers of the ESS MEBT as an example. For the ESS MEBT, $\lambda \sim 85.1$ cm, $\beta_0 \sim 0.0876$, and $\gamma_0 \sim 1$. Throughout the MEBT, the typical RMS value of x' and y' is ~5 mrad and the second order terms of x' and y' produce a ~0.12 degrees shift of the average phase over 1 m. Similarly, the typical RMS value of δ_p is ~0.2% and the second order term of δ_p produces ~0.2 degrees in 1 m.