Envelope Equations for a Superconducting RF Cavity

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Introduction

This note is not meant to be an original work for understanding RF cavities. It is intended to be a summary of the derivation of the envelope equations that describe that amplitude and phase response of a superconducting RF cavity that is driven by an external source.

RF Cavity Equivalent Circuit

The response of a cavity can be modeled by a parallel RLC circuit as shown in Figure 1. The inductor represents the magnetic energy stored in the cavity. The voltage across the inductor is given as:

$$v(t) = L\frac{di_L}{dt} \tag{1}$$

The resistor is inversely proportional to the power lost in the cavity walls. The voltage across the resistor is:

$$v(t) = Ri_R \tag{2}$$

The capacitor represents the electric energy stored in the cavity. The current though the capacitor is:

$$i_C = C\frac{dv}{dt} \tag{3}$$

The source current is divided amount the parallel LRC network:

$$i_s = i_L + i_R + i_C \tag{4}$$

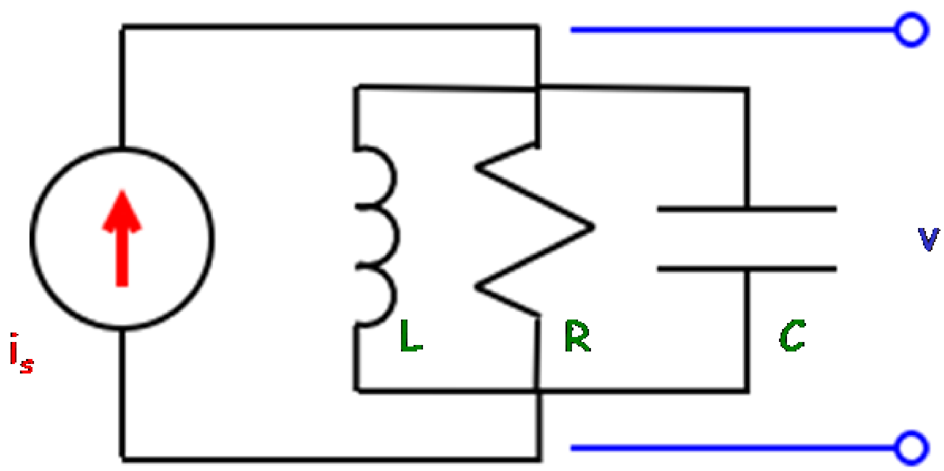

Figure 1. Equivalent Circuit for an RF Cavity

The following quantities are defined:

$$Q = \omega_o RC \tag{5}$$

$$\omega_o = \frac{1}{\sqrt{LC}} \tag{6}$$

Defined this way, the Q is the ratio of the energy stored to the energy lost in the circuit during one cycle. Note that this definition uses the energy lost in circuit and not just the cavity. An external circuit attached to the cavity will load the cavity circuit and dissipate power. The quantity ω_o is the resonant frequency of the cavity. Then:

$$L = \frac{R}{\omega_o Q} \tag{7}$$

$$C = \frac{Q}{\omega_o R} \tag{8}$$

The quantity R/Q is independent of R and is a function of the geometry only. The current flowing though the parallel LRC network becomes:

$$\frac{1}{{\omega_o}^2}\frac{d^2 i_L}{dt^2} + \frac{1}{\omega_o Q}\frac{di_L}{dt} + i_L = i_s \tag{9}$$

<u>Envelope Equations</u>

The solution to Eqn. 9 can be broken up into a fast varying and slow varying parts:

$$i_L(t) = Re\left\{I_L(t)e^{j\omega t}\right\} \tag{10}$$

$$i_s(t) = Re\left\{I_s(t)e^{j\omega t}\right\} \tag{11}$$

where $I_L(t)$ and $I_S(t)$ are slowly varying complex phasors. Substituting Eqns. 10 and 11 into Eqn. 9:

$$\frac{d^2 I_L}{dt^2} + \left(2j\omega + \frac{\omega_o}{Q}\right)\frac{dI_L}{dt} + j\frac{\omega\omega_o}{Q}I_L + ({\omega_o}^2 + \omega^2)I_L = {\omega_o}^2 I_s \tag{12}$$

For large ω, Eqn. 12 becomes:

$$2j\omega\frac{dI_L}{dt} + j\frac{\omega\omega_o}{Q}I_L + ({\omega_o}^2 + \omega^2)I_L \approx {\omega_o}^2 I_s \tag{13}$$

The cavity voltage is given as:

$$v(t) = \frac{R}{\omega_o Q}\frac{di_L}{dt} \tag{14}$$

The cavity voltage can also be separated into fast varying and slow varying parts:

$$v(t) = Re\left\{V(t)e^{j\omega t}\right\} \tag{15}$$

Substituting Eqn. 15 into Eq. 14:

$$V \approx j\omega \frac{R}{\omega_o Q} I_L \tag{16}$$

Equation 34 becomes:

$$\frac{2Q}{\omega_o}\frac{dV}{dt} + V - jQ\left(\frac{\omega_o}{\omega} - \frac{\omega}{\omega_o}\right)V = RI_s \tag{17}$$

In a steady state situation dV/dt=0:

$$V_\infty = \frac{RI_s}{1 - jQ\left(\frac{\omega_o}{\omega} - \frac{\omega}{\omega_o}\right)} \tag{18}$$

If $\omega \neq \omega_o$, the cavity is not operated on resonance (the cavity is de-tuned). When the cavity is de-tuned, the steady state cavity voltage and the generator are out of phase. The detuning angle is found from Eqn. 18:

$$tan(\varphi_D) = Q\left(\frac{\omega_o}{\omega} - \frac{\omega}{\omega_o}\right) \tag{19}$$

Equation 17 becomes:

$$\frac{2Q}{\omega_o}\frac{dV}{dt} + V - jtan(\varphi_D)V = RI_s \tag{20}$$

The cavity voltage can be separated into real and imaginary parts:

$$V = V_r + jV_i \tag{21}$$

Equation 20 becomes two coupled equations:

$$\frac{2Q}{\omega_o}\frac{dV_r}{dt} + V_r + tan(\varphi_D)V_i = RI_{s_r} \tag{22}$$

and:

$$\frac{2Q}{\omega_o}\frac{dV_i}{dt} + V_i - tan(\varphi_D)V_r = RI_{s_i} \tag{23}$$

The impulse response, h(t) to Equation 20 is derived from:

$$\frac{2Q}{\omega_o}\frac{dh(t,t')}{dt} + h(t,t') - jtan(\varphi_D)h(t,t') = R\tilde{q}\delta(t-t') \tag{24}$$

where q is an arbitrary charge to make the units correct and δ is the dirac delta function. The solution to Equation 24 is:

$$h(t,t') = \frac{\tilde{q}}{2}\omega_o \frac{R}{Q} e^{-\frac{\omega_0}{2Q}(t-t')} e^{j\frac{\omega_0}{2Q}tan(\varphi_D)(t-t')} \Theta(t-t') \tag{25}$$

where Θ(t-t’) is the Heavyside step function.