Dependence of cavity coupling on beam velocity

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Abstract

The coupling of a charged particle to the accelerating field in a cavity depends on the velocity of the particle, and the velocity for which the cavity has been designed. This note derives the functional form of this dependence for a wide range of accelerating cavities.

1 Introduction

The coupling, R/Q, of a charged particle to the accelerating field present in a cavity is dependent on the particle velocity, $v = \beta c$, as well as the spatial and temporal variation of the accelerating field.

2 Observed voltage

The total field "observed" by the particle will be equivalent to the integral of the field along the axis of the cavity, where the phase of the field is allowed to change in a way that matches the particle velocity.

$$V = E_1 \int_{-\frac{L}{2}}^{\frac{L}{2}} \cos\left(\frac{\omega}{\beta_0 c} z\right) \cos(\omega t)\, dz \tag{1}$$

$$V = E_1 \int_{-\frac{L}{2}}^{\frac{L}{2}} \cos\left(\frac{\omega}{\beta_0 c} z\right) \cos\left(\frac{\omega}{\beta c} z\right) dz \tag{2}$$

Thus the total voltage, V, observed by the particle depends on the field amplitude, E_1, the length, L, of the cavity, the angular frequency, ω, of the accelerating field, as well as the particle velocity, $v = \beta c$, and the velocity, $v_0 = \beta_0 c$, for which the cavity has been optimised.

The first cosine in this integral is the spatial variation of the accelerating field, which has been constructed as a single harmonic. In typical accelerating structures, many harmonics will be present, but this situation will be examined later. For the moment, it will be assumed that the fundamental spatial frequency is the only non-zero component.

The second cosine supplies the phase change of the RF due to the finite time taken for the particle to move through the field.

3 Integrating by parts

The integral in equation 2 may be approached by integrating by parts to give the following:

$$V = \left[E_1 \frac{\beta c}{\omega} \cos\left(\frac{\omega}{\beta_0 c} z\right) \sin\left(\frac{\omega}{\beta c} z\right) \right]_{-\frac{L}{2}}^{\frac{L}{2}} + \int_{-\frac{L}{2}}^{\frac{L}{2}} E_1 \frac{\beta c}{\omega} \sin\left(\frac{\omega}{\beta c} z\right) \frac{\omega}{\beta_0 c} \sin\left(\frac{\omega}{\beta_0 c} z\right) dz \tag{3}$$

The integral that appears on the right hand side of equation 3 may also subjected to integration by parts as follows:

$$V = \left[E_1 \frac{\beta c}{\omega} \cos\left(\frac{\omega}{\beta_0 c} z\right) \sin\left(\frac{\omega}{\beta c} z\right) \right]_{-\frac{L}{2}}^{\frac{L}{2}} - \left[E_1 \left(\frac{\beta c}{\omega}\right)^2 \left(\frac{\omega}{\beta_0 c}\right) \cos\left(\frac{\omega}{\beta c} z\right) \sin\left(\frac{\omega}{\beta_0 c} z\right) \right]_{-\frac{L}{2}}^{\frac{L}{2}} + \int_{-\frac{L}{2}}^{\frac{L}{2}} E_1 \frac{\beta^2}{\beta_0^2} \cos\left(\frac{\omega}{\beta c} z\right) \cos\left(\frac{\omega}{\beta_0 c} z\right) dz \tag{4}$$

Note that the integral that appears on the right hand side of equation 4 is identical to that in equation 2 scaled by a constant. Therefore, equation 2 may be solved by moving this integral to the left hand side of the equation, and dividing through by an appropriate constant.

$$V = 2 \frac{E_1}{k} \frac{\beta_0^2}{\beta_0^2 - \beta^2} \left(\beta \cos\left(\frac{\omega}{\beta_0 c} \frac{L}{2}\right) \sin\left(\frac{\omega}{\beta c} \frac{L}{2}\right) - \frac{\beta^2}{\beta_0} \cos\left(\frac{\omega}{\beta c} \frac{L}{2}\right) \sin\left(\frac{\omega}{\beta_0 c} \frac{L}{2}\right) \right) \tag{5}$$

Here the wavenumber, $k = \omega / c$. The fact that cosines are even functions, and sines are odd, has been used here to help simplify the result.

4 Cavity length

For a five–cell cavity designed for a particle velocity, $v_0 = \beta_0 c$, the total length of the cavity, $L = \frac{5}{2}\beta_0 \lambda$, where λ is the wavelength of the RF.

Thus the sines and cosines in equation 5 can be simplified as follows:

$$\frac{\omega}{c} \frac{L}{2} = k \frac{5\beta_0 \lambda}{4} = \frac{5\pi\beta_0}{2} \tag{6}$$

Equation 5 may then be reduced to:

$$V = 2 \frac{E_1}{k} \frac{\beta_0^2}{\beta_0^2 - \beta^2} \left(\beta \cos\left(\frac{5\pi}{2}\right) \sin\left(\frac{5\pi}{2} \frac{\beta_0}{\beta}\right) - \frac{\beta^2}{\beta_0} \cos\left(\frac{5\pi}{2} \frac{\beta_0}{\beta}\right) \sin\left(\frac{5\pi}{2}\right) \right) \tag{7}$$

Notice that the cosine in the first term goes to zero, thus leaving the final result:

$$V = -2\beta_0 \frac{E_1}{k} \frac{\beta^2}{\beta_0^2 - \beta^2} \cos\left(\frac{\beta_0}{\beta} \frac{5\pi}{2}\right) \tag{8}$$

5 Discussion

In the case where a cavity is accelerating a particle whose velocity matches that of the cavity (i.e. $\beta = \beta_0$) may seem problematic since the cosine term in equation 8 goes to zero, however note that the denominator, $\beta_0^2 - \beta^2$ also goes to zero.

In this limit,

$$V(\beta{=}\beta_0) = \frac{5\pi}{2} \beta_0 \frac{E_1}{k} \tag{9}$$

Also of interest are the zeros of the cosine on either side of the maximum, since these dictate the range of β for which the cavity will accelerate the beam. It can be seen from equation 8 that this range is:

$$\frac{5}{7} < \frac{\beta}{\beta_0} < \frac{5}{3} \tag{10}$$

6 Scaling of coupling for RF calculations

In calculating the required input power to a cavity, or the expected reflections from the input coupler during operation, it is important to remember that the coupling, R/Q, is normally given for the situation where $\beta = \beta_0$. Thus any calculations for a range of cavities should take the variation of the R/Q with β into account.

An additional function could be included in the calculations to apply the result from equation 8 into the analysis. In order to do this, it should be normalised by the $V(\beta{=}\beta_0)$ value shown in equation 9.

$$\left(\frac{R}{Q}\right)_\beta = \left(\frac{R}{Q}\right)_{\beta_0} \left[\frac{4}{5\pi} \frac{\beta^2}{\beta_0^2 - \beta^2} \cos\left(\frac{\beta_0}{\beta} \frac{5\pi}{2}\right)\right]^2 \tag{11}$$

Notice that this equation contains the square of the normalised voltage from equation 8 due to the fact that the coupling, R/Q, is proportional to the square of the observed voltage, V.

Equation 11 has been plotted in figure 1 for a five-cell β_0=0.90 cavity. Note that the fact that the optimal coupling lies at a higher β than β_0 is a well known phenomena, and gives reassurance that the derivation presented so far is correct.

7 Differing number of cells

The result in equation 8 was for the case of a five-cell cavity, however it is also interesting to generalise this to the case of an arbitrary number, N, of cells.

In this case, the substitution given in equation 6 changes to:

$$\frac{\omega}{c} \frac{L}{2} = k \frac{N\beta_0\lambda}{4} = \frac{N\pi\beta_0}{2} \tag{12}$$

Equation 7 then becomes:

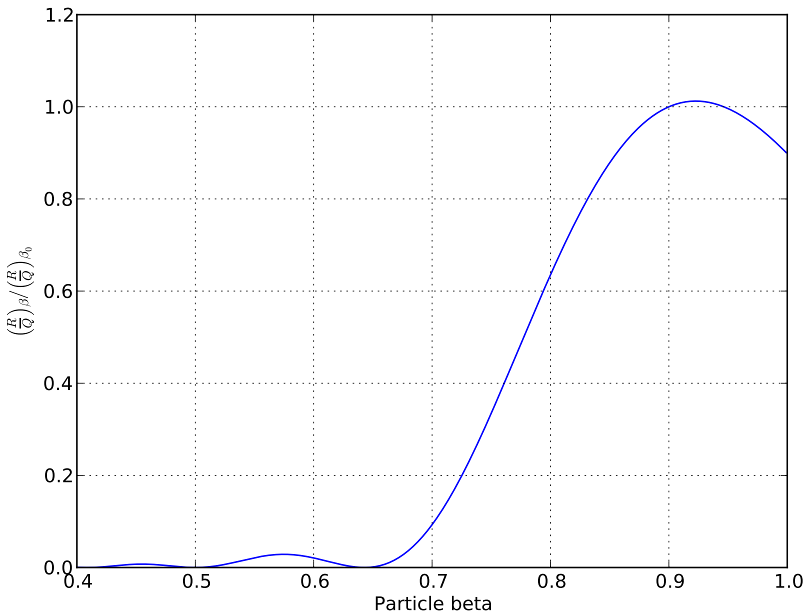

Figure 1: R/Q dependence on β for a β_0=0.90 cavity.

$$V = 2\frac{E_1}{k}\frac{\beta_0^2}{\beta_0^2-\beta^2}\left(\beta\cos\left(\frac{N\pi}{2}\right)\sin\left(\frac{N\pi}{2}\frac{\beta_0}{\beta}\right) - \frac{\beta^2}{\beta_0}\cos\left(\frac{N\pi}{2}\frac{\beta_0}{\beta}\right)\sin\left(\frac{N\pi}{2}\right)\right) \tag{13}$$

Equation 13 may be simplified in a way that depends on whether N is odd or even.

If N is odd, then the cosine in the first term vanishes.

$$V = -2\beta_0\frac{E_1}{k}\frac{\beta^2}{\beta_0^2-\beta^2}\cos\left(\frac{N\pi}{2}\frac{\beta_0}{\beta}\right)\sin\left(\frac{N\pi}{2}\right) \tag{14}$$

This result agrees with equation 8.

In the limit of $\beta = \beta_0$,

$$V(\beta{=}\beta_0) = \frac{N\pi}{2}\beta_0\frac{E_1}{k} \tag{15}$$

Note that the sine term from equation 14 no longer appears since it is multiplied by an identical term that appears through the use of l'Hopital's rule to evaluate the limit. Since this sine term can only equal ± 1, its square will always be equal to unity.

If N is even then the sine in the second term vanishes.

$$V = 2\frac{E_1}{k}\frac{\beta\beta_0^2}{\beta_0^2-\beta^2}\sin\left(\frac{N\pi}{2}\frac{\beta_0}{\beta}\right)\cos\left(\frac{N\pi}{2}\right) \tag{16}$$

In the limit of $\beta = \beta_0$, this goes to:

$$V(\beta{=}\beta_0) = \frac{N\pi}{2}\beta_0\frac{E_1}{k} \tag{17}$$

Note that the cosine term from equation 16 no longer appears in equation 17 since it is forced to unity in the same way as the sine term from equation 14 when evaluating equation 15.

Note that equations 15 and 17 are identical as required. Also reassuring is the linear dependence on the number of cells, N.

The velocity bandwidth for both odd & even N is then:

$$\frac{N}{N+2} < \frac{\beta}{\beta_0} < \frac{N}{N-2} \tag{18}$$

The reduction of the velocity acceptance of the cavity with increasing N can be seen from equation 18.

8 Multiple spatial harmonics

In many cases, it is not sufficient to model the z variation of the accelerating field as a single spatial mode, and so this result should be generalised to the case of a summation of multiple modes. Equation 2 is altered as follows.

$$V = \sum_{n=1}^{\infty} E_n \int_{-\frac{L}{2}}^{\frac{L}{2}} \cos\left(\frac{n\omega}{\beta_0 c}z\right)\cos\left(\frac{\omega}{\beta c}z\right) dz \tag{19}$$

Much of the analysis proceeds as for equation 2, but with additional factors of n and n^2 appearing after the second integration, and with the result appearing inside the summation over n.

Equation 5 becomes,

$$\begin{aligned} V = \sum_{n=1}^{\infty} 2\frac{E_n}{k}\frac{\beta_0^2}{\beta_0^2 - n^2\beta^2}\left(\beta\cos\left(\frac{n\omega}{\beta_0 c}\frac{L}{2}\right)\sin\left(\frac{\omega}{\beta c}\frac{L}{2}\right) - \right. \\ \left. \frac{n\beta^2}{\beta_0}\cos\left(\frac{\omega}{\beta c}\frac{L}{2}\right)\sin\left(\frac{n\omega}{\beta_0 c}\frac{L}{2}\right)\right) \end{aligned} \tag{20}$$

Performing the substitution given by equation 6 leads to,

$$\begin{aligned} V = \sum_{n=1}^{\infty} 2\frac{E_n}{k}\frac{\beta_0^2}{\beta_0^2 - n^2\beta^2}\left(\beta\cos\left(\frac{5n\pi}{2}\right)\sin\left(\frac{5\pi}{2}\frac{\beta_0}{\beta}\right) - \right. \\ \left. \frac{n\beta^2}{\beta_0}\cos\left(\frac{5\pi}{2}\frac{\beta_0}{\beta}\right)\sin\left(\frac{5n\pi}{2}\right)\right) \end{aligned} \tag{21}$$

8.1 Fourier approximation to a square wave

To more closely model the spatial distribution of the accelerating field within, for example, a spoke cavity, a square wave is a closer approximation than a pure sinusoid, and so the following will examine the implications on the coupling of adding in an additional harmonic:

- $E_1 = 1.125E_1$
- $E_3 = 1.125E_1/9$

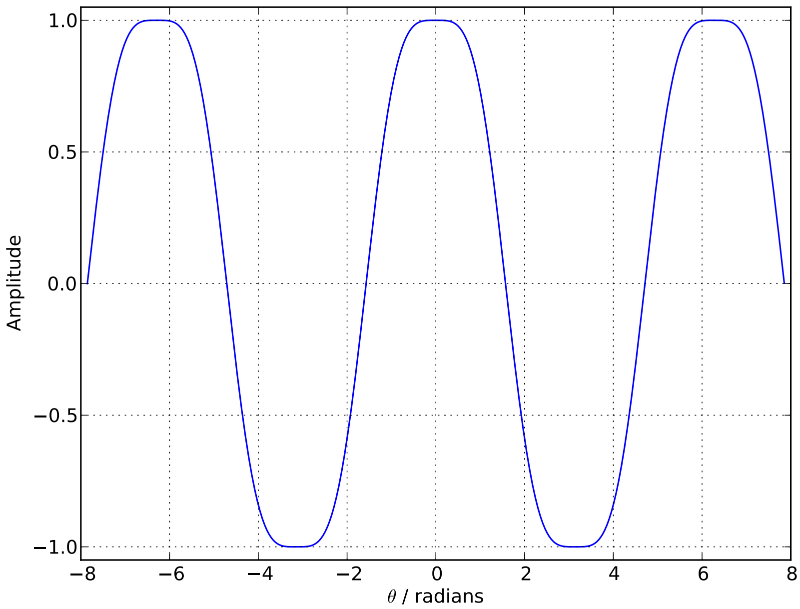

Figure 2: Voltage profile calculated from the given E_1 and E_3.

Where the factor of 1.125 scales to achieve the correct scaling of the maximum field in the cavity. All other harmonics remain at zero amplitude. This profile is shown in figure 2.

$$\begin{aligned} V = 2.25\frac{E_1}{k}\frac{\beta_0^2}{\beta_0^2-\beta^2}\left(\beta\cos\left(\frac{5\pi}{2}\right)\sin\left(\frac{5\pi}{2}\frac{\beta_0}{\beta}\right)-\right.\\ \left.\frac{\beta^2}{\beta_0}\cos\left(\frac{5\pi}{2}\frac{\beta_0}{\beta}\right)\sin\left(\frac{5\pi}{2}\right)\right)+\\ \frac{2.25}{9}\frac{E_1}{k}\frac{\beta_0^2}{\beta_0^2-9\beta^2}\left(\beta\cos\left(\frac{15\pi}{2}\right)\sin\left(\frac{5\pi}{2}\frac{\beta_0}{\beta}\right)-\right.\\ \left.\frac{3\beta^2}{\beta_0}\cos\left(\frac{5\pi}{2}\frac{\beta_0}{\beta}\right)\sin\left(\frac{15\pi}{2}\right)\right) \end{aligned} \tag{22}$$

Simplifying,

$$V = 2.25\beta_0\beta^2\frac{E_1}{k}\cos\left(\frac{5\pi}{2}\frac{\beta_0}{\beta}\right)\left[\frac{1}{\beta_0^2-\beta^2}-\frac{1}{3\left(\beta_0^2-9\beta^2\right)}\right] \tag{23}$$

It may be observed that the second term in the bracket in equation 23 only adds a very small change to the final result, and it may be ignored.

Thus the field defined by E_1 and E_3 is almost identical to the field defined purely by E_1. This is a consequence of the fact that the act of integrating the product of this field with the cosine term due to the beam's velocity has the effect of extracting the component of the field with the same frequency, i.e. E_1.

Therefore, a cavity with the field structure defined above has no advantage in terms of the velocity acceptance, since this is still defined by the number of cells as in equation 18.

The advantage of this field structure is the 12.5% increase in the total coupling due to the magnitude of the E_1 component used to produce this variation. To create this increase without the addition of the E_3 component would require an increase in the surface fields within the cavity, thereby increasing the risk of cavity failure.

9 Conclusions

The dependence of the beam–cavity coupling on the velocity, $v = \beta c$, of the beam has been determined. The number of cells, as well as the number of spatial harmonics used to correctly specify the z distribution of the accelerating field, and the design velocity, $v = \beta_0 c$, of the cavity, are free parameters to this calculation, making it applicable to a wide range of cavity designs.

Note that all the calculations in this note assume a π phase advance of the field from cell-to-cell, however adjusting the derivation to alternative phase advances is a matter of changing the substitution given in equation 6 to the appropriate value.